AD-775 735

ENERGY SPECTRA OF THE OCEAN SURFACE: II. INTERACTION WITH SURFACE CURRENT

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Prepared for:

Rome Air Development Center
Defense Advanced Research Projects Agency

October 1973

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

AD775 735

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER RADC-TR-74-15 / 2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ENERGY SPECTRA OF THE OCEAN SURFACE: II. INTERACTION WITH SURFACE CURRENT	5. TYPE OF REPORT & PERIOD COVERED Semi-Annual
	6. PERFORMING ORG. REPORT NUMBER PD-73-037
7. AUTHOR(s) Kenneth M. Watson, Bruce J. West, Bruce I. Cohen	8. CONTRACT OR GRANT NUMBER(s) F30602-72-C-0494
9. PERFORMING ORGANIZATION NAME AND ADDRESS Physical Dynamics, Inc. P.O. Box 1069 Berkeley, Ca. 94701	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PEC 62301E Job Order: 16490402
11. CONTROLLING OFFICE NAME AND ADDRESS Defense Advanced Research Projects Agency 1400 Wilson Blvd. Arlington, Va. 22209	12. REPORT DATE October 1973
	13. NUMBER OF PAGES 66
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) RADC/OCSE ATTN: L. Strauss Griffiss AFB, N.Y. 13441	15. SECURITY CLASS. (of this report) UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Ocean waves
Modal expansion
Surface waves
Surface currents
Linear interactions
Surface spectra
Internal waves
Eigenmodes
Tank and ocean experiments

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The linear interaction equations between surface gravity waves and a surface current is reviewed. These equations are cast in the form of coupled eigenmode equations which are first order in time and can be solved numerically. The solutions to these coupled equations provide one with a view as to how a specified spectrum of surface waves is modified by a surface current. This analysis extends that in RADC-TR-73-74 (PD-72-030). It is found that the interaction regimes of tank experiments and ocean experiments can be quite

different so that information obtained from the former does not necessarily provide information about the latter.

PD-73-037

ENERGY SPECTRA OF THE OCEAN SURFACE:
II. INTERACTION WITH SURFACE CURRENT

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Contractor: Physical Dynamics, Incorporated
Contract Number: F30602-72-C-0494
Effective Date of Contract: 1 May 1972
Contract Expiration Date: 31 December 1973
Amount of Contract: $91,818.00
Program Code Number: 2E20

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Approved for public release;
distribution unlimited.

This research was supported by the Defense Advanced Research Projects Agency of the Department of Defense and was monitored by Leonard Strauss RADC (OCSE), GAFB, NY 13441 under Contract F30602-72-C-0494.

PUBLICATION REVIEW

This technical report has been reviewed and is approved.

Leonard Strauss
RADC Project Engineer

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1. INTRODUCTION

The first report in this series [Watson, West and Thomson, (I), PD-72-030 (RADC-TR-73-74)] was concerned with the development of a formalism which would allow a direct calculation of the non-linear interaction among surface gravity waves. The technique used was to expand the ocean surface in a series of modes, i.e., a finite Fourier series, and express Bernoulli's equation and the kinematic boundary condition at the ocean surface in terms of interacting modes. These modal equations formed a system of rate equations with quartic and triplet mode couplings giving the non-linear interactions. These equations were then transformed to normal coordinates, i.e., to the eigenmodes of the linear system, and integrated numerically on a computer. A calculation[1,2] with simulated initial conditions compared favorably with the Benjamin-Feir (BF) experiment (1967).

In this second report we wish to extend the above analysis to include the interaction of the surface gravity waves with a surface current. We will treat the surface current as being generated by a "massive" internal wave, i.e., an internal wave which does not react to the surface gravity waves [see Thomson and West, PD-72-023 (RADC-TR-72-280)], and consider only the linear interaction between the current and surface gravity waves. The consequences of relaxing this assumption,

i.e., allowing the internal wave to change, is being investigated elsewhere (Watson, West, Cohen, PD-73-032). Section 2 reviews the construction of the linear equation and obtains results consistent with those of M. Milder (1972) and indicates an error by Zachariasen (1972) pertaining to the surface on which the equations are evaluated.

In Section 3, a translating surface current is expanded in harmonics and introduced into the linear equations, which are expressed in terms of modes. The equations are written in two dimensions, since the added complexity over the one dimensional equations is not great. The eigenmode equations close to resonance are rather simple, a given mode $\vec{k}$ being coupled only to its neighboring modes $\vec{k}\pm\vec{K}$, where $\vec{K}$ is the fundamental wavenumber of the internal wave. A hermitian representation of the interaction equation can be approximated near resonance and an interesting form of this equation is derived in two dimensions.

The two dimensional nature of the equations is not exploited in this report, but a discussion indicating the possible direction of future work is given. In the one dimensional form, the expressions are compared to those developed by Rosenbluth (1971), where an inconsistency in the level of approximation is found. This inconsistency arises from the additional terms uncovered in the analysis of Zacharisen (1972). A modal expression exact to order 1/N is derived and the solution in terms of convolutions with discrete Greens functions is formulated.

The modal equations are solved numerically in Section 4 for a number of cases. The first calculation employs initial conditions which simulate the experiment of Lewis (1971). In this experiment, an internal wave was generated at a fluid interface near the bottom of a tank. An infinitesimal surface wave was mechanically generated and the modulation of the surface height and slope recorded. We calculate the modulation as a function of time, as well as the development of the surface spectrum and compare the results with the experimental data.

A second calculation is presented in which the parameters are characteristic of the ocean environment rather than those of a tank. The case of a single (dominant) surface wave in an unsaturated surface spectrum interacting with an internal wave is considered. The results are discussed in the context of the linear theory developed in Section 3 and the effect of nonlinear tranfer of energy between surface waves is also considered. The distinction between the interaction regimes for tank experiments and the ocean experiments is also discussed. The important quantity is determined to be the relative magnitude of the correction to the dispersion relation for the eigenmodes and the coupling coefficients of the surface waves to internal waves.

2. LINEAR WAVE EQUATION

This section is essentially a review of the derivation of the linear wave equation and may be omitted by those familiar with its construction. In this development we consider the surface gravity waves to be infinitesimal and to be interacting with a non-uniform surface current. The surface current may be generated by a non-dispersive internal wave, a swell passing through the region of interest or by a number of other mechanisms. For the present we will not be concerned with how the surface current is produced, only that it is present. This analysis compliments that in PD 72-023 (RADC-TR-72-280), i.e., the linear wave equation being constructed includes terms which were considered negligible previously and found by Zachariasen (1972) to be important.

We assume the ocean to be both homogeneous and irrotational so that we can use a potential description of the velocity field, i.e., since $\nabla \times \vec{u} = 0$ we can write $\vec{u} = \nabla\Phi$. The ocean is also assumed to be incompressible ($\nabla \cdot \vec{u} = 0$) so that the velocity potential satisfies Laplace's equation,

$$\nabla^2\Phi = 0 \quad . \tag{2.1}$$

The integral of the momentum equation yields Bernoulli's equation at the ocean surface

$$\Phi_t + \frac{1}{2}\nabla\Phi\cdot\nabla\Phi + gz = 0 \qquad \text{at} \quad z = \zeta \tag{2.2}$$

with the condition that hydrostatic equilibrium must prevail at infinity and $z = \zeta$ is the free surface of the ocean. The velocity potential must also satisfy the kinematic boundary condition at the ocean surface, i.e., the rate of increase in the wave height following a fluid element is the vertical component of the fluid velocity,

$$\frac{d\zeta}{dt} = \frac{\partial\Phi}{\partial z} \qquad \text{at} \quad z = \zeta \tag{2.3}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \nabla\Phi\cdot\nabla \tag{2.4}$$

is the Eulerian derivative.

The linearized forms of the dynamic equations may be obtained by separating the effects of the infinitesimal surface waves and surface current. We do this by writing the velocity potential and surface displacement as,

$$\Phi = \Psi + \phi \tag{2.5}$$

and

$$\zeta = \chi + h \tag{2.6}$$

where Ψ and χ are associated with the current and ϕ and h are associated with the infinitesimal surface waves. In the approximation we are considering, the effect of the surface waves on

the current is negligible. Therefore, Ψ and χ satisfy the free surface equations independent of the surface height (h) and velocity potential (ϕ), i.e.,

$$\Psi_t + \frac{1}{2} \nabla\Psi \cdot \nabla\Psi + g\chi = 0 \quad \text{at} \quad z = \chi \tag{2.7}$$

and

$$\chi_t + \nabla_s\Psi \cdot \nabla_s\chi = \frac{\partial\Psi}{\partial z} \quad \text{at} \quad z = \chi \quad . \tag{2.8}$$

Note that Equations (2.7) and (2.8) refer to the unperturbed ocean surface, that is, unperturbed by the surface waves.

To determine the interaction between the gravity waves and current, we substitute Equations (2.5) and (2.6) into Equation (2.2) to obtain,

$$\phi_t + \frac{1}{2} \nabla\phi \cdot \nabla\phi + \nabla\phi \cdot \nabla\Psi + gh = -\left(\Psi_t + \frac{1}{2} \nabla\Psi \cdot \nabla\Psi + g\chi\right) \quad \text{at} \quad z = \chi + h \tag{2.9}$$

The right hand side of Equation (2.9) would apparently vanish using Equation (2.7) except that the two expressions are not evaluated at the same boundary. It is necessary, therefore, to expand Equation (2.9) about the ambient surface, i.e., the surrounding region on which there are no gravity waves. The expansions for the velocity potential are as follows:

$$\nabla\Psi \Big|_{z = \chi + h} \approx \left\{\nabla\Psi + h \frac{\partial}{\partial z} (\nabla\Psi)\right\} \Big|_{z = \chi} \tag{2.10}$$

and

$$\Psi_t \Big|_{z=\chi+h} \approx \left\{ \Psi_t + h \frac{\partial}{\partial z} (\Psi_t) \right\} \Big|_{z=\chi} \qquad (2.11)$$

in which we have kept only terms linear in the surface wave height (h).

We define the current as follows:

$$\nabla\Psi = \vec{U} + V_s \hat{k} \qquad (2.12)$$

where $\hat{k}$ is a unit vector in the vertical direction. The horizontal current $\vec{U}$ is significant regardless of the generating mechanism, the vertical component V_s, however, is not. For example, if the current is produced by a non-dispersive internal wave, then $|U| >> |V_s|$ so that only the horizontal component need be retained in Equation (2.12). If, however, the current is produced by a passing swell, the vertical component of the current may be comparable to the horizontal, so that both must be considered. The particular generating mechanism will therefore determine the form of the current used in the analysis.

Using the definition of the current [Equation (2.12)] and the expansions given by Equations (2.10) and (2.11) in the interaction equation [Equation (2.9)] we obtain,

$$\phi_t + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \left[g + \frac{\partial V_s}{\partial t} + (\nabla\phi + \nabla\Psi)\cdot\nabla V_s\right]h + \nabla\Psi\cdot\nabla\phi = 0 \quad \text{at} \quad z = \chi \quad . \tag{2.13}$$

If we now restrict our analysis to the linear case, i.e., linear in both ϕ and h, we reduce Equation (2.13) to

$$\phi_t + \nabla\Psi\cdot\nabla\phi + \left(g + \frac{dV_s}{dt}\right)h = 0 \quad \text{at} \quad z = \chi \quad . \tag{2.14}$$

We note that in addition to the standard linear interaction term between the surface waves and current ($\nabla\Psi\cdot\nabla\phi$) there is a second interaction due to the vertical component of the current field. This modification of the gravitational acceleration has been discussed recently by M. Milder[10].

In determining the linear form of the kinematic boundary condition [Equation (2.3)], we encounter the same difficulty as with the interaction equation, i.e., going from the perturbed to the ambient surface. We expand the terms in Equation (2.3), just as before, and obtain

$$V_s\Big|_{z=\chi+h} \approx \left\{V_s + h\frac{\partial V_s}{\partial z}\right\}\Big|_{z=\chi} \quad ; \quad \left(V_s = \frac{\partial\Psi}{\partial z}\right) \tag{2.15}$$

to terms linear in h. Using Equation (2.15) along with the other expansions [Equations (2.10) and (2.11)] and the kinematic boundary condition for the current [Equation (2.8)] in Equation (2.3) yields,

$$\left(\frac{\partial}{\partial t} + \nabla_s\phi\cdot\nabla_s + \nabla_s\Psi\cdot\nabla_s + \nabla V_s{}_s\cdot\nabla_s\right)h = \frac{\partial\phi}{\partial z} + h\,\frac{\partial V_s}{\partial z} - \nabla_s\phi\cdot\nabla_s X - \nabla V_s{}_s\cdot\nabla_s X$$

$$\text{at} \quad z = X$$

which to terms linear in ϕ and h becomes

$$\frac{dh}{dt} = \frac{\partial\phi}{\partial z} + h\,\frac{\partial V_s}{\partial z} - \nabla_s\phi\cdot\nabla_s X \qquad \text{at} \quad z = X \quad . \tag{2.16}$$

The incompressibility condition applied to the surface current yields the relation

$$\frac{\partial V_s}{\partial z} = \frac{\partial^2\Psi}{\partial z^2} = -\nabla_s^2\Psi = -\nabla_s\cdot\vec{U}$$

which when substituted into Equation (2.16) results in,

$$\left(\frac{d}{dt} + \nabla_s\cdot\vec{U}\right)h = \frac{\partial\phi}{\partial z} - \nabla_s\phi\cdot\nabla_s X \qquad \text{at} \quad z = X \quad . \tag{2.17}$$

We may replace the gradient of the ambient surface in Equation (2.17) by taking the gradient of Equation (2.7) and using the definition of the current to write

$$\begin{aligned}\nabla_s X &= -\frac{1}{g}\,\nabla_s\left[\Psi_t + \tfrac{1}{2}(\vec{U}^2 + V_s^2)\right] \\ &= -\frac{1}{g}\left[\frac{d\vec{U}}{dt} + V_s\,\nabla_s V_s\right]\end{aligned} \tag{2.18}$$

Substituting Equation (2.18) into Equation (2.17) yields,

$$\left(\frac{d}{dt} + \nabla_s\cdot\vec{U}\right)h = \phi_z + \frac{1}{g}\,\frac{d\vec{U}}{dt}\cdot\nabla_s\phi \tag{2.19}$$

Neglecting terms quadratic and higher, except when they involve a coupling of the surface waves to the surface current, we obtain from Equations (2.14) and (2.19),

$$\phi_t + \nabla\Psi\cdot\nabla\phi + gh = 0 \tag{2.20a}$$

$$h_t + \nabla_s\Psi\cdot\nabla_s h + (\nabla_s\cdot\vec{U})h = \phi_z + \frac{1}{g}\frac{\partial\vec{U}}{\partial t}\cdot\nabla_s\phi \tag{2.20b}$$

evaluated at $z=\chi$. Although these equations have the same structure in one dimension as those derived by Zachariasen (1972), they are evaluated at $z=\chi$ and not $z=0$ as were his expressions. The small terms obtained by further expanding the set (2.20) about $z=0$ are found to be of importance.

Milder (1972)[6] has shown that the simplicity of these equations can be maintained by noting that the right-hand side of Equation (2.20b) can be written as,

$$\frac{\partial\phi}{\partial z} - \frac{\partial\chi}{\partial x}\frac{\partial\phi}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial\phi}{\partial n} \tag{2.21}$$

in a single horizontal dimension x, where s is the distance along the surface $z = \chi(x,t)$ and $\frac{\partial}{\partial n}$ is the derivative normal to that surface. Milder then shows that because the velocity potential satisfies the Laplacian, in two dimensions,

$$\nabla^2\phi = \left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial z}\right)\phi = 0 \tag{2.22}$$

that

$$\frac{\partial s}{\partial x}\frac{\partial \phi}{\partial n} = -i\,\frac{\partial \phi}{\partial x} \tag{2.23}$$

and Equation (2.20) reduces to,

$$\begin{aligned} &\frac{d}{dt}\,\phi + gh = 0 \\ &\frac{d}{dt}\,h + U_x h = -i\,\phi_x \end{aligned} \tag{2.24}$$

where the x derivatives are constrained to the surface $z = \chi(x,t)$ and $\frac{d}{dt}$ is the Lagrangian derivative, $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + U\,\frac{\partial}{\partial x}$. This analysis may be extended to two dimensions, resulting in the loss of the $\frac{\partial \vec{U}}{\partial t}$ term in Equation (2.20b) (see Appendix in Watson and West, PD-73-048).

3. MODAL ANALYSIS WITH A HARMONIC SURFACE CURRENT

This section is concerned with the interaction between surface gravity waves and a current distribution

$$\vec{U} = U(x,t)\,\vec{e}_x$$

where

$$U(x,t) = \sum_K U_K \cos K(x - c_I t) \tag{3.1}$$

and $\vec{e}_x$ is a unit vector in the direction of propagation. The approach used is that developed in Section I in which eigenmode equations were constructed to represent the interaction of gravity waves with gravity waves. We will extend that analysis to include the interaction of surface gravity waves to surface currents prescribed by Equation (3.1). The linear solution to the gravity wave-surface current interaction problem has been obtained by a number of investigators using a variety of approaches; e.g., Rosenbluth (1971), Ko (1971) and Zachariasen (1972). In this section we will indicate in the context of the eigenmode model how these various calculations differ in their level of approximation. In the next section, we will extend the calculation into the nonlinear regime.

The first order equations for the velocity potential and surface elevation derived in the preceding sections

[Equations (2.20) and (2.21)] easily accommodates the current distribution defined by Equation (3.1). At the unperturbed surface, the interaction may be written in two dimensions as

$$\phi_t + gh = - \sum_K U_K \phi_x \cos K\xi$$

and (3.2)

$$h_t - \phi_z = - \sum_K U_K \left\{ h_x \cos K\xi - Kh \sin K\xi \right\}$$

where $\xi = x - c_I t$ and c_I is the phase velocity of the internal wave packet. Because the velocity potential satisfies Laplace's equation, we may write the two-dimensional Fourier transforms of ϕ and the surface elevation as

$$\phi(\vec{r},z,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}} e^{|\vec{k}|z} \phi_{\vec{k}}(t) \, d^2k$$

and (3.3)

$$h(\vec{r},t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{r}} h_{\vec{k}}(t) \, d^2k$$

and $\vec{r} = (x,y)$. Although the wavenumbers are continuous in Equation (3.3), we will find that the dynamic equations restrict their values to a discrete set. The resulting expressions will, therefore, conform to the definitions introduced in Section I.

The interaction equations may be expressed in terms of the mode amplitudes as

$$\int_{-\infty}^{\infty}\left[e^{|\vec{k}|z}\,\dot{\phi}_{\vec{k}} + gh_{\vec{k}} + i|\vec{k}|\sum_K U_K\,\phi_{\vec{k}}\cos K\xi\right]e^{i\vec{k}\cdot\vec{r}}\,d^2k = 0$$

and

$$\int_{-\infty}^{\infty}\left[\dot{h}_{\vec{k}} - |\vec{k}|\,e^{|\vec{k}|z}\,\phi_{\vec{k}} + i|\vec{k}|\sum_K U_K\,h_{\vec{k}}\cos K\xi\right]e^{i\vec{k}\cdot\vec{r}}\,d^2k$$

$$= \int_{-\infty}^{\infty}\sum_K K\,U_K\,h_{\vec{\ell}}\,\sin K\xi\,e^{i\vec{\ell}\cdot\vec{r}}\,d^2\ell \quad . \tag{3.4}$$

We recall that the linearized equations have been obtained at the surface $z = \chi$. The exponential terms in Equation (3.4) may therefore be set equal to unity since $|\vec{k}|\chi << 1$ for the surface gravity waves of interest. If we multiply Equation (3.4) by exp $[-i\vec{k}'\vec{r}]$ and integrate over the surface plane $(\vec{r})$, we obtain

$$\dot{\phi}_{\vec{k}} + gh_{\vec{k}} = -\sum_K \frac{i}{2} U_K\left[(k_x - K)\,\phi_{\vec{k}-\vec{K}}\,e^{-iKc_It} + (k_x + K)\phi_{\vec{k}+\vec{K}}\,e^{iKc_It}\right]$$

and (3.5a)

$$\dot{h}_{\vec{k}} - |\vec{k}|\,\phi_{\vec{k}} = -\sum_K \frac{i}{2} U_K\left\{e^{-iKc_It}\left[(k_x - K)\,h_{\vec{k}-\vec{K}} + Kh_{\vec{k}-\vec{K}}\right]\right.$$

$$\left. + e^{iKc_It}\left[(k_x + K)\,h_{\vec{k}+\vec{K}} - Kh_{\vec{k}+\vec{K}}\right]\right\} \tag{3.5}$$

as the linearized mode coupled equations to second order in the interaction.

3A. The Eigenmode Expansion

We now introduce the eigenmode variables constructed in Section I:

$$b^{(\pm)}(\vec{k}) = A(k)\left\{\phi_{\vec{k}} \pm i\,\frac{\omega(k)}{|\vec{k}|}\,h_{\vec{k}}\right\} \tag{3.6}$$

where $\omega(k)$ is given by the linear dispersion relation ($= \sqrt{g|\vec{k}|}$) and $A(k)$ is chosen to be

$$A(k) = \frac{|\vec{k}|^2}{\sqrt{2\pi}\,\omega(k)} . \tag{3.7}$$

By taking the sum and difference of Equations (3.5), we obtain the eigenmode equations

$$i\dot{b}^{(\pm)}(\vec{k}) \mp \omega(k)b^{(\pm)}(\vec{k}) = A(k)\left\{iF_1(\vec{k}) \pm \frac{\omega(k)}{|\vec{k}|}\,F_2(\vec{k})\right\} \tag{3.8}$$

where $F_1(\vec{k})$ and $F_2(\vec{k})$ are defined by the right-hand sides of Equations (3.5a) and (3.5b), respectively.

We may rewrite Equation (3.8) in the more explicit form

$$\frac{\partial b^{(+)}(\vec{k})}{\partial t} + i\omega(k)b^{(+)}(\vec{k}) = -\frac{i}{4}\sum_K U_K\,|\vec{k}|\,\frac{k_x-K}{|\vec{k}-\vec{K}|}\,e^{-iKc_I t}$$

$$\cdot\left[\left\{1 + \frac{K}{k_x-K} + \frac{\omega(k)}{\omega(k-K)}\right\} b^{(+)}(\vec{k}-\vec{K}) - \left\{1 + \frac{K}{k_x-K} - \frac{\omega(k)}{\omega(k-K)}\right\} b^{(-)}(\vec{k}-\vec{K})\right]$$

$$- \frac{i}{4} \sum_K U_K |\vec{k}| \frac{k_x+K}{|\vec{k}+\vec{K}|} e^{iKc_I t}$$

$$\cdot \left[\left\{ 1 - \frac{K}{k_x+K} + \frac{\omega(k)}{\omega(k+K)} \right\} b^{(+)}(\vec{k}+\vec{K}) \right.$$

$$\left. - \left\{ 1 - \frac{K}{k_x+K} - \frac{\omega(k)}{\omega(k+K)} \right\} b^{(-)}(\vec{k}+\vec{K}) \right] \quad (3.9a)$$

and the second coupled equation

$$\frac{\partial b^{(-)}(\vec{k})}{\partial t} - i\,\omega(k)\, b^{(-)}(\vec{k}) = - \frac{i}{4} \sum_K U_K\, |\vec{k}|\, \frac{k_x-K}{|\vec{k}-\vec{K}|} e^{-iKc_I t}$$

$$\cdot \left[\left\{ 1 + \frac{K}{k_x-K} + \frac{\omega(k)}{\omega(k-K)} \right\} b^{(-)}(\vec{k}-\vec{K}) \right.$$

$$\left. - \left\{ 1 + \frac{K}{k_x-K} - \frac{\omega(k)}{\omega(k-K)} \right\} b^{(+)}(\vec{k}-\vec{K}) \right]$$

$$- \frac{i}{4} \sum_K U_K\, |\vec{k}|\, \frac{k_x+K}{|\vec{k}+\vec{K}|} e^{iKc_I t}$$

$$\cdot \left[\left\{ 1 - \frac{K}{k_x+K} + \frac{\omega(k)}{\omega(k+K)} \right\} b^{(-)}(\vec{k}-\vec{K}) \right.$$

$$\left. - \left\{ 1 - \frac{K}{k_x+K} - \frac{\omega(k)}{\omega(k+K)} \right\} b^{(+)}(\vec{k}+\vec{K}) \right] \quad (3.9b)$$

where we have used $\Omega = c_I K$ as the frequency of the internal wave. Examining the structure of the coupling coefficients in Equation (3.9) makes it clear that $b^{(+)}$ and $b^{(-)}$ are only weakly coupled, since $|\vec{k}|, k_x >> K$. It is only necessary, therefore, to discuss the equations for either $b^{(+)}$ or $b^{(-)}$ but not both.

We may remove the explicit time dependence from Equation (3.9) by introducing the variables*

$$c^{(\pm)}(\vec{k}) = b^{(\pm)}(\vec{k})\, e^{ik_x c_I t} \quad . \tag{3.10}$$

If we introduce Equation (3.10) into Equation (3.9a) and use the decoupling approximation, we obtain the equation for right travelling waves

$$\begin{aligned} i\dot{c}(\vec{k}) &= [\omega(k) - k_x c_I]\, c(\vec{k}) \\ &+ \sum_K \frac{U_K}{4} |\vec{k}| \left\{ \frac{k_x - K}{|\vec{k}-\vec{K}|} \left[1 + \frac{K}{k_x - K} + \frac{\omega(k)}{\omega(k-K)} \right] c(\vec{k}-\vec{K}) \right. \\ &+ \left. \frac{k_x + K}{|\vec{k}+\vec{K}|} \left[1 - \frac{K}{k_x + K} + \frac{\omega(k)}{\omega(k+K)} \right] c(\vec{k}+\vec{K}) \right\} \end{aligned} \tag{3.11}$$

where we have deleted the extraneous (+) superscript. A second and perhaps stronger justification for the form of Equation (3.11) is based on the notion of a resonant interaction between the surface gravity waves. This notion was

*This transformation was suggested by Kenneth Case and the resulting Hermitian equations are explored elsewhere, Watson, West and Case, 1973 (III).

used in I to delete terms whose frequencies could not be matched in the third order interaction. A similar argument could be constructed above so that only terms where the frequency difference

$$\Delta_{k\mp K} = \omega(k) - \omega(k\mp K) \mp \Omega \tag{3.12}$$

can vanish contribute strongly to Equation (3.9). These are just the terms kept in Equation (3.11).

Equation (3.11) gives the rate of change in the k-th gravity mode slope in two dimensions produced by the interaction with an arbitrary travelling surface current. Since the interaction is assumed to be linear, i.e., we used Equations (2.20) and (2.21), the dynamic equation may be added directly to the nonlinear expression developed in I. The composite equation in one dimension is

$$ic(k) = [\omega(k) - kc_I]c(k) + \sum_K \frac{U_K}{4} k[a_{k-K}c(k-K) + a_{k+K}c(k+K)]$$

$$- \sum_{\ell+p=k+n} \Gamma_{\ell p}^{kn}\, c(\ell)\; c(p)\; c^*(n) \tag{3.13}$$

where the coupling coefficients $\Gamma_{\ell p}^{kn}$ are given in I, the $a_{k\mp K}$ are given in Equation (3.11) and we have neglected wind, viscosity and surface tension.

3B. Approximate Two-Dimensional Equation

First let us note that, while $\vec{k}$ is a continuous variable, Equation (3.11) is really a discrete set. Thus for a given $c(\vec{k})$, we need only consider the equations for $c(\vec{k}')$ where $\vec{k}' = \vec{k} + \sum_{k} m_K \vec{K}$ with m_K being a positive or negative integer. To obtain a Hermitian form of Equation (3.11), we restrict our attention to the case of a single internal wave of wavenumber K; then

$$
\begin{aligned}
f(\vec{k}) &= \frac{U_K}{4} |\vec{k}| \frac{k_x - K}{|\vec{k}-\vec{K}|} \left\{ 1 + \frac{K}{k_x - K} + \frac{\omega(k)}{\omega(k-K)} \right\} \\
g(\vec{k}) &= \frac{U_K}{4} |\vec{k}| \frac{k_x + K}{|\vec{k}-\vec{K}|} \left\{ 1 - \frac{K}{k_x + K} + \frac{(k)}{\omega(k+K)} \right\}
\end{aligned}
\tag{3.13}
$$

and introduce

$$\rho(\vec{k}) = \omega(k) - k_x c_I \quad . \tag{3.14}$$

We consider the condition where $|\vec{k}| >> K$ which is applicable to the ocean situation. In this limit, Equation (3.11) reduces to

$$i \frac{\partial c(\vec{k})}{\partial t} = [\omega(k) - k_x c_I]\, c(\vec{k}) + \frac{U_K}{2} |\vec{k}| \left\{ c(\vec{k}+\vec{K}) + c(\vec{k}-\vec{K}) \right\} . \tag{3.15}$$

The change of variable

$$c(\vec{k}) = \sqrt{|\vec{k}|}\; \psi(\vec{k})$$

in Equation (3.11) could also be used to obtain

$$i \frac{\partial \psi(\vec{k})}{\partial t} = [\omega(k) - k_x c_I]\, \psi(\vec{k}) + \sum_K \frac{U_K}{2} \left\{ a_K(\vec{k}+\vec{K})\; \psi(\vec{k}+\vec{K}) + a_K(\vec{k})\; \psi(\vec{k}-\vec{K}) \right\} \tag{3.16}$$

where

$$a_K(\vec{k}) = \sqrt{|\vec{k}|\,|\vec{k} - \vec{K}|} \quad . \tag{3.17}$$

which reduces to Equation (3.15) for $|\vec{k}| >> |\vec{K}|$. Equation (3.16) gives the interaction of the gravity waves with an internal wave spectrum in a Hermitian representation.

To obtain an alternate expression for the two-dimensional problem, we focus attention on Equation (3.15) for the case of a single internal wave. We note that we have been concerned with the wavenumber region $|\vec{k}| >> K$, so that we may employ the following approximation

$$\frac{1}{2}\left\{c(\vec{k}+\vec{K}) + c(\vec{k}-\vec{K})\right\} \approx c(\vec{k}) + \frac{1}{2} K^2 \nabla^2_{\vec{k}} c(\vec{k}) + \ldots \qquad (3.18)$$

where $\nabla_{\vec{k}}$ is the gradient operator in $\vec{k}$-space. If in addition to Equation (3.18) we look at the eigenvalue problem

$$c(\vec{k}) = \phi_\lambda(\vec{k})\, e^{-i\lambda t} \qquad (3.19)$$

Equation (3.15) becomes

$$\nabla^2_{\vec{k}} \phi_\lambda(\vec{k}) + \frac{2}{K^2 U_k}\left[\frac{\omega'(k) - \lambda}{|\vec{k}|}\right]\phi_\lambda(\vec{k}) = 0 \qquad (3.20)$$

where

$$\omega'(k) = \omega(k) + |\vec{k}|\, U_k - k_x c_I \quad . \qquad (3.21)$$

The quantity $\omega'(k)$ is the frequency of the surface wave in the coordinate system translating with the phase velocity of the internal wave and is therefore a constant.

The efficacy of using the expansion given by Equation (3.18) is discussed elsewhere (III) where an exact one-dimensional eigenvalue problem is solved and compared with that obtained using the expansion. The eigenvalues and eigenfunctions of the approximate expression are found to be quite close to those given by the exact solutions. We are therefore confident that the general behavior of the solution to Equation (3.15) may be determined by examining the solutions to Equation (3.20).

3C. Approximate One-Dimensional Equation

Let us from here on focus our attention on the one-dimensional form of Equation (3.15). In one dimension $\vec{k} = (k_x, 0)$ and since the interaction equations form a discrete set we use the wavenumber $k_x = nK$, $n = 0, 1, \ldots N-1, N, N+1, \ldots$. For a single internal wave of wavenumber K, we may rewrite Equation (3.15) as

$$i\dot{c}_n = [\omega(n) - n\Omega]c_n + \frac{1}{2} U_o nK \left\{c_{n+1} + c_{n-1}\right\} \tag{3.22}$$

where $\Omega(=Kc_I)$ is the frequency of the internal wave. We recall that the strongly interacting surface modes are those for which the frequency resonance condition given by Equation (3.12) holds. If we assume that the central surface mode is located by an integer $n >> 1$ and $\omega(n) >> \Omega$, then we may find a simple approximate expression for Equation (3.22).

(i) Expansion in Terms of Block Waves

We introduce a new variable which is shifted in phase from the central mode $n = N$

$$a_v = c_n e^{i[\omega - N\Omega]t} \tag{3.23}$$

into Equation (3.22) to obtain

$$[\omega - \omega(n) - N\Omega + n\Omega]a_v = \frac{1}{2} U_o K n [a_{v+1} + a_{v-1}] \quad . \tag{3.24}$$

Note that the new expansion coefficients $[a_v]$ are constants in time and v measures the deviation of the wavenumber from the central mode, i.e., $n = N + v$.

We now proceed to expand $\omega(N+v)$ about $\omega(N)$ and determine the value of Ω in terms of N. We select the frequency ω to be

$$\omega = \omega(N) + \delta\omega \tag{3.25}$$

so that Equation (3.24) becomes

$$\left\{\delta\omega + \frac{1}{2}\left(\frac{v}{N}\right)^2 \sqrt{gNK}\right\} a_v = \frac{U_o K}{2} (N+v)(a_{v+1} + a_{v-1})$$

or expanding $(N+v)^{-1}$, we have

$$a_{v+1} + a_{v-1} - U_o^{-1} \left[\frac{2\delta\omega}{k_N} + \frac{1}{2}\frac{\Omega}{K}\left(\frac{v}{N}\right)^2\right] a_v \approx 0 \tag{3.26}$$

where Ω has been set equal to $\frac{1}{2}\sqrt{gK/N}$. We note that Equation (3.26) is the expression obtained by Rosenbluth (1971) for the interaction of discrete modes when the expansion set for the surface elevation and velocity potential are Block Waves. It might also be mentioned that the expansion given by Equation (3.18) in the two-dimensional case is equivalent to adding zero to Equation (3.26) in the form $2a_v - 2a_v$ and taking the limit of continuous v as done by Rosenbluth. We also note that although terms to order $(v/N)^2$ have been maintained in Equation (3.26), terms of lower order have previously

been deleted, i.e., order v/N. These are the terms

$\frac{K}{k_x \cdot K}$ in the coupling coefficients [Equation (3.13)] which arise from the current gradient terms in Equation (2.21) and were first discussed by Zachariasen (1972).

Because of the inconsistency in the order of terms maintained in Equation (3.26), we take a slightly different approach than that of Rosenbluth. We introduce the variable

$$a_v(t) = c_n(t)\, e^{i[\omega(N)-N\Omega]t}\, e^{iv\frac{\pi}{2}} \tag{3.27}$$

into the one-dimensional form of Equation (3.11) to obtain

$$\begin{aligned} \dot{a}_v(t) = &- i\,[\omega(n) - \omega(N) - v\Omega]\, a_v(t) \\ &+ \frac{U_o}{4}\, nK \left[\left(1 + \frac{1}{n-1} + \frac{\omega(n)}{\omega(n-1)}\right) a_{v-1} \right. \\ &\left. - \left(1 - \frac{1}{n+1} + \frac{\omega(n)}{\omega(n+1)}\right) a_{v+1}\right] \quad . \end{aligned} \tag{3.28}$$

If we expand $\omega(N+v)$ about $\omega(N)$ and equate Ω with $\frac{1}{2}\sqrt{gK/N}$ then

$$\omega(N+v) \approx \omega(N) + v\Omega \;;\quad n = N + v \tag{3.29a}$$

and

$$\frac{\omega(n)}{\omega(n\mp1)} \approx 1 \pm \frac{1}{2n} \quad . \tag{3.29b}$$

Substituting Equation (3.29) into Equation (3.28) results in the expression

$$\dot{a}_v(t) = \frac{U_o K}{2} n (a_{v-1} - a_{v+1}) + \frac{3U_o K}{8} (a_{v-1} + a_{v+1}) \tag{3.30}$$

which is exact to order $\frac{v}{N}$. The second term in Equation (3.30) is of order $\frac{1}{N}$ smaller than the first and invites a perturbation approach to the problem.

(ii) Perturbation Solution of Equation (3.30)

Let us consider a solution to Equation (3.30) of the form

$$a_v(t) = a_v^{(o)}(t) + \delta a_v(t) \tag{3.31}$$

where $\delta a_v(t)$ is of order $\frac{v}{N}$ smaller than $a_v^{(o)}(t)$. If we scale the time in Equation (3.30) as

$$\tau = U_o KNt \tag{3.32}$$

and introduce the perturbation expression Equation (3.31), we obtain

$$\frac{d}{d\tau} a_v^{(o)}(t) = \frac{1}{2} \left[a_{v-1}^{(o)}(t) - a_{v+1}^{(o)}(t) \right] \tag{3.33a}$$

and

$$\frac{d}{d\tau}\,\delta a_v(t) = \frac{1}{2}\left[\delta a_{v-1} - \delta a_{v+1}\right] + \frac{3}{8N}\left[a_{v-1}^{(o)} + a_{v+1}^{(o)}\right]$$

$$+ \frac{1}{2}\,\frac{v}{N}\left[a_{v-1}^{(o)} - a_{v+1}^{(o)}\right] \qquad (3.33b)$$

where we have deleted terms of order $1/N^2$. Equations (3.33a and b) are correct to order $1/N$.

It is clear that Equation (3.33a) is the recursion relation for Bessel functions. The solution to our first order equation can therefore be written as

$$a_v^{(o)}(t) = \sum_{\ell=-\infty}^{\infty} d_\ell^{(o)}\, J_{\ell-v}(t) \qquad (3.34)$$

where $J_m(\tau)$ is the Bessel function of the first kind of order m. The expansion coefficients $(d_\ell^{(o)})$ are determined by the fact that only $m = 0$ Bessel functions are non-zero at time $\tau = 0$ and are selected to satisfy the initial conditions for the problem.

Using the first order solution given by Equation (3.34) and the additional recursion relation for Bessel functions

$$\frac{2n}{\tau}\, J_n(t) = J_{n-1}(\tau) + J_{n+1}(t) \qquad (3.35)$$

in Equation (3.33b) we obtain

$$\frac{d}{d\tau}\,\delta a_v = \frac{1}{2}\left\{\delta a_{v-1} - \delta a_{v+1}\right\} + \frac{3}{4N}\sum_{\ell=-\infty}^{\infty} d_\ell^{(o)}\left\{v\,\frac{dJ_{\ell-v}}{d\tau} + \frac{\ell-v}{\tau}\,J_{\ell-v}\right\}. \tag{3.36}$$

Equation (3.36) is clearly a linear equation in δa_v with a driving force given by the sum over Bessel functions. We may solve Equation (3.36) by rewriting the equation as

$$\frac{d}{d\tau}\,\delta a_v(\tau) - \frac{1}{2}\left[\delta a_{v-1}(\tau) - \delta a_{v+1}(\tau)\right] = F_v(\tau) \tag{3.37}$$

where

$$F_v(\tau) = \frac{3}{4N}\sum_{\ell=-\infty}^{\infty} d_\ell^{(o)}\left\{v\,\frac{dJ_{\ell-v}(t)}{d\tau} + \frac{\ell-v}{\tau}\,J_{\ell-v}(t)\right\}. \tag{3.38}$$

If we introduce the Laplace transforms

$$\mathcal{L}[\delta a_v(\tau)] = \delta a_v(s) \equiv \int_0^\infty e^{-s\tau}\,\delta a_v(\tau)d\tau$$

and (3.39)

$$\mathcal{L}[F_v(\tau)] = F_v(s) \equiv \int_0^\infty e^{-s\tau}\,F_v(\tau)d\tau$$

into Equation (3.37), we obtain

$$s\,\delta a_v(s) - \frac{1}{2}\left[\delta a_{v-1}(s) - \delta a_{v+1}(s)\right] = F_v(s) + \delta a_v(o) \tag{3.40}$$

after integrating $\mathcal{L}\left[\frac{d\delta a_v}{d\tau}\right]$ by parts.

We observe that Equation (3.40) may be integrated by using the discrete Green's function $g(v,s)$ which satisfies the equation

$$sg(v,s) - \frac{1}{2}\left[g(v-1,s) - g(v+1,s)\right] = \delta_{vo} \quad . \tag{3.41}$$

The solution to Equation (3.41) is given by

$$g(v,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-ivKx}\,dx}{s-i\sin Kx} \tag{3.42}$$

which may be verified by direct substitution. The formal solution to Equation (3.40) may now be written as

$$\delta a_v(s) = \sum_{v'} g(v-v', s)\left[F_{v'}(s) + \delta a_{v'}(o)\right] \quad . \tag{3.43}$$

The inverse of $\delta a_v(s)$ may be obtained by recalling

$$\mathcal{L}^{-1}\left\{\mathcal{L}[f_1(\tau)]\,\mathcal{L}[f_2(\tau)]\right\} = \int_o^{\tau} f_1(\tau')\, f_2(\tau-\tau')\, d\tau' \tag{3.44}$$

where $\mathcal{L}^{-1}\{f\}$ is the inverse Laplace transform of f. Using Equation (3.44) on Equation (3.43) yields

$$\delta a_v(\tau) = \sum_{v'} \int_o^{\tau} g(v-v', \tau-\tau')[F_v(\tau') + \delta a_{v'}(o)\,\delta(\tau')]\, d\tau' \quad . \tag{3.45}$$

The Green's function in Equation (3.45) is given by

$$g(v-v', \tau-\tau') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i(v-v')Kx} e^{i(\tau-\tau')\sin Kx} dx \; . \tag{3.46}$$

Because we are dealing with a discrete set of states, we approximate the limits on this integral by $\pm \frac{1}{K}$, which is the periodic condition imposed on the surface waves over the wavelength of the internal wave. In this approximation, Equation (3.46) yields

$$g(v-v', \tau-\tau') \approx J_{v-v'}(\tau-\tau') \tag{3.47}$$

so that Equation (3.43) becomes

$$\delta a_v(\tau) = \sum_{v'} g(v-v', \tau) \; \delta a_{v'}(o) + \frac{3}{4N} \sum_{v', \ell=-\infty}^{\infty} d_\ell^{(o)} \int_o^\tau \left| v' \frac{dJ_{\ell-v'}}{d\tau'} \right.$$

$$\left. + \frac{\ell-v'}{\tau'} J_{\ell-v'} \right| J_{v-v'}(\tau-\tau') d\tau'$$

as the general solution to Equation (3.33b) with $d_\ell^{(o)}$ given by the initial conditions for the solution to (3.33a).

The surface elevation is given by the discrete form of the one-dimensional Fourier transform in Equation (3.3). This, together with the defining equations [(3.6) and (3.10)] yields

$$h(x,t) = -\frac{i}{2}\sum_n \frac{1}{k_n}\left\{c_n^{(+)} e^{ik_n(x-c_I t)} - c_n^{(-)} e^{ik_n(x-c_I t)}\right\} .$$

(3.49)

We note that for a situation with only right travelling waves, i.e., predominantly in the direction of the wind, $c_n^{(-)} \equiv 0$ for $k_n > 0$ and $c_{-n}^{(-)} \equiv (c_n^{(+)})^*$ so that

$$h(x,t) = \sum_{n>0} \frac{r_n}{k_n} \sin\left[k_n(x-c_I t) + \theta_n\right] \tag{3.50}$$

where

$$c_n^{(+)}(t) \equiv r_n e^{i\theta_n} . \tag{3.51}$$

Using Equations (3.27), (3.34) and (3.48), we have for the slope variable

$$c_n(t) = \left\{a_v^{(o)}(t) + \delta a_v(t)\right\} \exp\left[- i(\omega(N) - N\Omega)t - iv\frac{\pi}{2}\right]$$

$$= \sum_{\ell=-\infty}^{\infty} d_\ell^{(o)} \left\{J_{\ell-v}(\tau) + \frac{3}{4N}\sum_{v'=-\infty}^{\infty} \int_o^\tau \left\{v' \frac{dJ_{\ell-v'}(\tau')}{d\tau'}\right.\right.$$

$$\left.\left. + \frac{\ell-v'}{\tau'} J_{\ell-v'}(\tau')\right\} d\tau' J_{v-v'}(\tau-\tau')\right\}$$

$$\times \exp\left[- i(\omega(N) - N\Omega)t - iv\frac{\pi}{2}\right] \tag{3.52}$$

where we have set $\delta a_v(o) = 0$. Inserting Equation (3.52) into the expansion for the surface elevation results in

$$h(x,t) = \sum_{N+v>0} \sum_{\ell=-\infty}^{\infty} \frac{N+\ell}{N+v} h_{N+\ell}(o) \left\{ J_{\ell-v}(\tau) + \frac{1}{N} \sum_{v'=-\infty}^{\infty} \int_{o}^{\tau} g(v-v',\tau-\tau')F_{v'}(\tau')d\tau' \right\} \times \sin\left[\theta_N + k_v\xi - v\frac{\pi}{2}\right] \tag{3.53}$$

where

$$\theta_N = k_N x - \omega(N)t \tag{3.54}$$

and

$$k_v\xi = vK(x - c_I t) \quad .$$

Equation (3.53) is a superposition of travelling wave with time dependent amplitudes.

If we expand the sine terms in Equation (3.53), we find the surface elevation can be written as

$$h(x,t) = |G(x,t)| \cos\left[\theta_N - \arctan(G_I/G_R)\right] \tag{3.55}$$

where

$$G(x,t) = \sum_{N+v>0} \frac{1}{N+v} \sum_{\ell=-\infty}^{\infty} (N+\ell)\, h_{N+\ell}^{(o)} \left\{ J_{\ell-v}(\tau) + \frac{1}{N} \sum_{v'=-\infty}^{\infty} \int_{o}^{\tau} g(v-v',\tau-\tau')\, F_{v'}(\tau')\, d\tau' \times \exp\, i\left(k_v\xi - v\frac{\pi}{2}\right) \right\} \quad . \tag{3.56}$$

The function G(x,t), therefore, determines the modulation of the central mode k_N(= NK). We may express the envelope function in terms of $c_n(t)$ rather than the surface elevation so that

$$G(x,t) = \sum_{N+v>0} \frac{c_{N+v}(t)}{k_{N+v}} e^{i[k_v \xi + N\Omega t]} . \tag{3.57}$$

If we use the slope variables introduced in I, i.e.,

$$q_n^{(+)} = c_n^{(+)} e^{i[\omega(n) - n\Omega]t} \tag{3.58}$$

then we would have from Equation (3.57)

$$G(x,t) = \sum_{N+v>0} \frac{q_{N+v}^{(+)}(t)}{k_{N+v}} e^{ik_v(x - c_g t)} \tag{3.59}$$

where

$$c_g = [\omega(N+v) - \omega(N)]/k_v \tag{3.60}$$

is the group velocity of the surface wave packet.

The modulation function for the surface slope $\left(\frac{dh}{dx}\right)$ may be found in a similar manner to be

$$G_s(x,t) = \sum_{N+v>0} \sum_{\ell=-\infty}^{\infty} (N+\ell) h_{N+\ell}(o) \left\{ J_{\ell-v}(\tau) + \frac{3}{4N} \sum_{v'=-\infty}^{\infty} \int_0^{\tau} g(v-v', \tau-\tau') F_{v'}(\tau') d\tau' \right\} \left[e^{i\left(K\xi - \frac{\pi}{2}\right)} \right]^v . \tag{3.61}$$

If we introduce the variable

$$z = e^{i(K\xi + \frac{\pi}{2})}$$

into Equation (3.61), then we have an expansion in terms of z^{ν}. This result can be used for formal manipulation of the interaction equation when the solution is expressed as a generating function,

$$g(\xi,t) = \sum_{\nu} a_{\nu} z^{\nu} \quad .$$

4. NUMERICAL CALCULATIONS

4A. Introduction

In this section we wish to explore the relative effects on the redistribution of energy within a given initial spectrum by both the mechanism of nonlinear interaction of surface gravity waves and the interaction of these waves with a prescribed surface current. To do this we use the full nonlinear expression shown in Section 3, Equation (3.13). Our discussion will center on two calculations, both giving some insight into the relative importance of the interactive mechanisms. The first calculation will be that of an impulsively applied harmonic surface current interacting with a single surface mode. The growth of the spectrum as a function of time will be discussed, as well as the nonlinear effects on the surface modulation. This simplified picture will be compared with the linear calculations of the Lewis experiment as an indication of the veracity of the numerical techniques used in the integration. The second calculation replaces the single surface mode of the preceding case with a discrete surface spectrum interacting with an internal wave with typical oceanographic parameters.

4B. Linear Surface Modes

In this calculation, we wish to focus on the interaction between a prescribed surface current and surface gravity waves. To isolate this interaction from the nonlinear interaction between surface gravity waves, we consider small amplitude surface waves, so that the nonlinear interaction is inhibited. The analysis in Section 3 is appropriate for calculations in which the parameters of the problem are characteristic of the ocean environment. It is inappropriate, however, for conditions found in typical tank experiments. This situation becomes clear if we examine the relative size of the interaction coefficients to the doppler shift in the eigenmode frequency.

We recall that the frequency of the (N+v) sideband was approximated in Section 3 by

$$\omega(N+v) \approx \omega(N) + v\Omega \quad . \tag{4.1}$$

The next order term in this expansion, which was neglected in Section 3, would be $v^2\Omega/4N$, where $\Omega = \omega(N)/2N$ and the central wavenumber is given by $k_N = NK$. Typical values for ocean and tank experiments are

Ocean	**Tank**
$\Omega \sim 10^{-2}\ \mathrm{sec}^{-1}$	$\Omega \sim 1\ \mathrm{sec}^{-1}$
$K \sim 2\times10^{-4}\ \mathrm{cm}^{-1}$	$K \sim 0.05\ \mathrm{cm}^{-1}$
$N \sim 500$	$N \sim 8$
$U_o \sim 2$ cm/sec	$U_o \sim 10^{-2}$ cm/sec
$v^2\ \Omega/4N \sim 5\times10^{-6}\ v^2$	$v^2\ \Omega/4N \sim 0.03\ v^2$
$U_o k \sim 0.1\ \mathrm{sec}^{-1}$	$U_o k \sim 2\times10^{-3}\ \mathrm{sec}^{-1}$

In the ocean environment the interaction coefficient is typically 10^4 larger than the doppler shift; however, in tanks the doppler shift is typically an order of magnitude larger than the coupling coefficient. The tank is therefore in a separate region of the experimental range than is the ocean.

To see this result in detail, we consider the interaction equation for the surface gravity waves with a surface current generated by a single internal wave with the frequency correction term included

$$\dot{a}_v(t) = i\, v^2 \frac{\Omega}{4N} a_v(t) + \frac{1}{2} U_o N K \left\{ a_{v-1}(t) - a_{v+1}(t) \right\} . \tag{4.2}$$

The terms neglected in Equation (4.2) are an order of magnitude smaller than the coupling coefficient which in turn are an order of magnitude smaller than the frequency correction term.

Let us consider the problem of an impulsively-applied harmonic current to a single small amplitude surface wave, i.e., at time t = 0 we "turn on" the interaction between a single surface mode and internal mode both of infinite spatial extent. If we concern ourselves with only the growth of the first two side bands, we obtain the coupled system

$$\dot{a}_1(t) = i\,\frac{\Omega}{4N}\,a_1(t) + \frac{1}{2}\,K\,U_o\,N\,a_o(t)$$

$$\dot{a}_o(t) = \frac{1}{2}\,N\,K\,U_o\,\left\{a_{-1}(t) - a_1(t)\right\}$$

$$\dot{a}_{-1}(t) = i\,\frac{\Omega}{4N}\,a_{-1}(t) - \frac{1}{2}\,N\,K\,U_o\,a_o(t) \tag{4.3}$$

where $k_N = NK$ is the wavenumber of the initial wave. We may combine the equations in (4.3) to obtain

$$\ddot{a}_1(t) - i\,\frac{\Omega}{4N}\,\dot{a}_1(t) + \left(\frac{NKU_o}{2}\right)^2 a_1(t) = \left(\frac{NKU_o}{2}\right)^2 a_{-1}(t)$$

$$\ddot{a}_{-1}(t) - i\,\frac{\Omega}{4N}\,\dot{a}_{-1}(t) + \left(\frac{NKU_o}{2}\right)^2 a_{-1}(t) = \left(\frac{NKU_o}{2}\right)^2 a_1(t)\ . \tag{4.4}$$

The coefficient of the forcing function is of the order 10^{-5} sec^{-2} and the solution to the system may be obtained to a good approximation by setting the right-hand side equal to zero, especially for the linear case, yielding

$$a_{\pm 1}(t) = \frac{\dot{a}_{\pm 1}(o)}{\omega} e^{-i\frac{\Omega}{8N}t} \sin\omega t; \quad \omega = \frac{1}{2}\sqrt{(NKU_o)^2 + \left(\frac{\Omega}{4N}\right)^2} \tag{4.5}$$

when $a_{\pm 1}(o) = 0$.

The modulation function for the surface elevation is given by Equation (3.57) which in terms of the a_ν's becomes

$$G(x,t) = \sum_{\nu=-1}^{+1} (-i)^\nu \frac{a_\nu(t)}{k_{N+\nu}} e^{i\nu K\xi} e^{i\theta_N} \tag{4.6}$$

where

$$\theta_N = k_N x - \omega(N)t$$

and (4.7)

$$\xi = x - c_I t \quad .$$

In terms of the solutions for the impulsively applied internal wave, we have

$$G(x,t) = \frac{a_o(o)\, e^{i\theta_N}}{k_N} \left\{ 1 - \frac{NKU_o}{\omega} \sin\omega t \left[\frac{1}{N}\sin K\xi + \cos K\xi \right] e^{-i\frac{\Omega}{8N}t} \right\} \tag{4.8}$$

for the modulation of the initial wave. For the early time solution, i.e., $\omega t << 1$, we can write

$$G(x,t) \approx h(o)\, e^{i\theta N} \Big\{ 1 - KU_o t \left[\sin K\xi + \frac{\omega t}{8} \cos K\xi \right]$$

$$- iNU_o Kt \left[\cos K\xi - \frac{\omega}{8N^2}\, t \sin K\xi \right] \Big\} \tag{4.9}$$

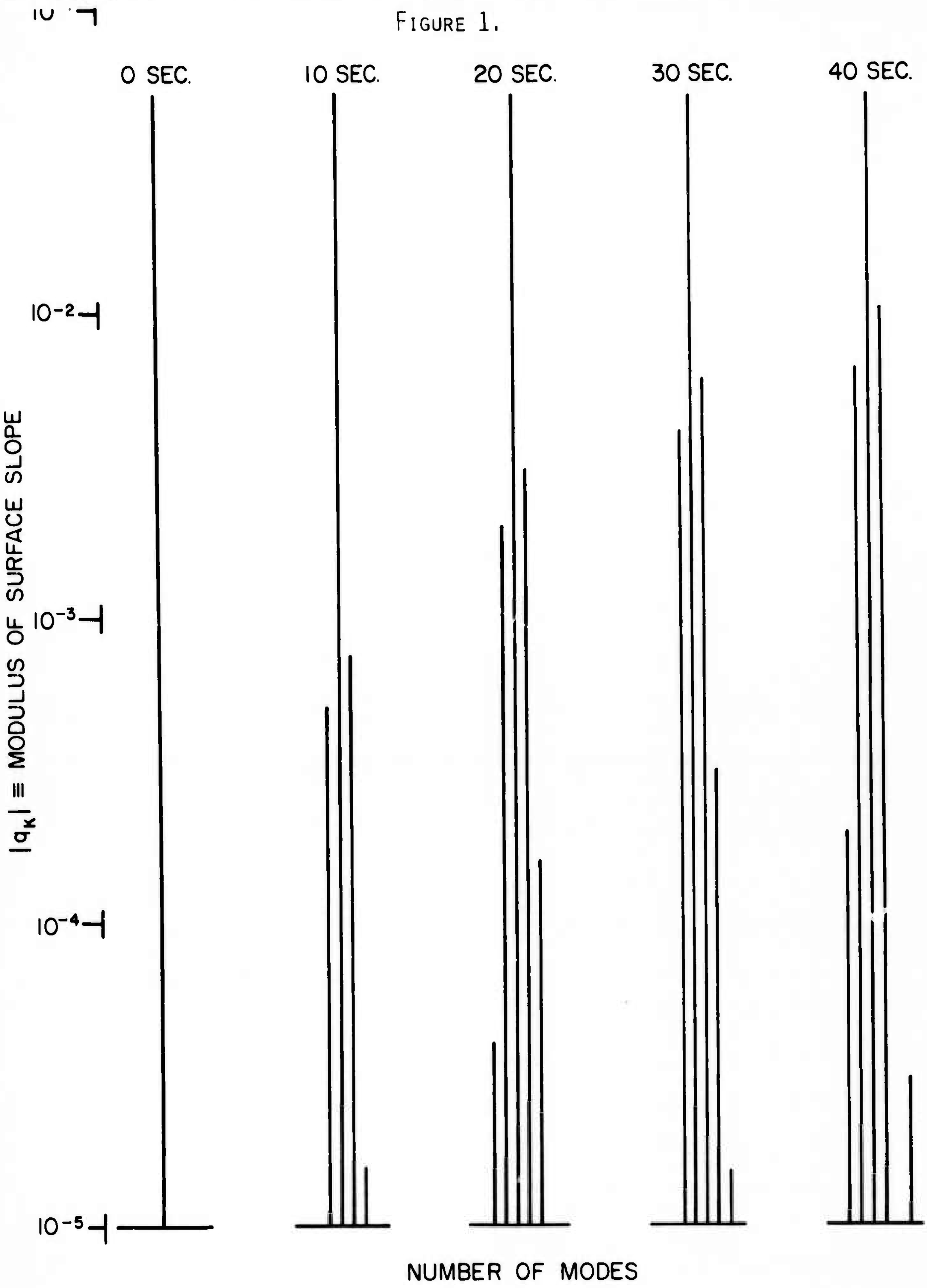
since $\omega \sim 0.02 \text{ sec}^{-1}$ so that $t << 50$ sec. We can also write

$$|G(x,t)| \approx h(o) e^{i\theta N} \Big\{ 1 - U_o Kt \left[\sin K\xi + \frac{\omega t}{8} \cos K\xi \right] \Big\} \tag{4.10}$$

which agrees with Zachariasen's calculation for the initial value problem. Equation (4.10) may now be used to check the numerical calculation at early times.

For the numerical calculation, we selected the wave-number of the surface mode to be $k_o = 0.412 \text{ cm}^{-1}$ and internal mode number $K = 0.0515 \text{ cm}^{-1}$. These values were chosen to be in the range used by Lewis (1971) in his interaction experiment. For an internal wave of phase speed of $c_I = 24.39$ cm/sec, the surface and internal wave are in resonance. Figure (1) depicts the discrete spectrum of surface waves at five different times; the "0 sec" graph illustrating the initial conditions for our problem. The initial slope $|q_{k_o}| = 0.0524$ is that

FIGURE 1.


0 SEC.
10 SEC.
20 SEC.
30 SEC.
40 SEC.
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$|q_K| \equiv$ MODULUS OF SURFACE SLOPE
NUMBER OF MODES

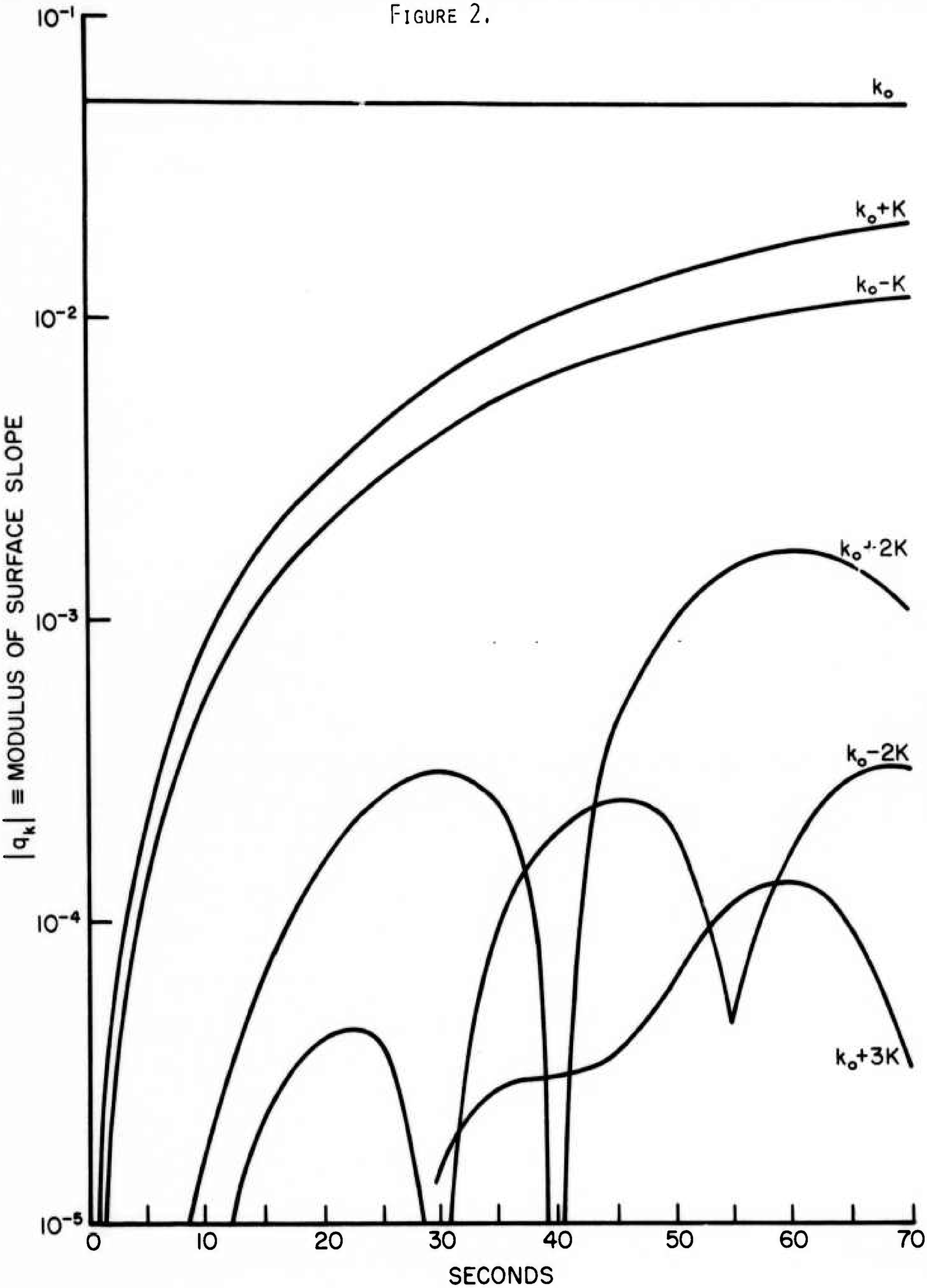
of the mechanically generated surface wave in the Lewis experiment. The spacing between modes is given by the internal mode number $\Delta k = K$.

The spectral "snapshots" in Figure (1) indicate the persistence of a rather narrow spectrum. Only the sidebands of the primary wave have gained appreciably in energy after a time interval corresponding to twelve cycles of the central mode. The other modes seem to acquire and lose energy, but not to steadily increase. This is seen more clearly in the following figure.

In Figure (2) we indicate the continuous growth of the spectral mode slopes in time. It is clear that three of the nine modes in the calculation dominate the characteristics of the spectrum. In hindsight then, the inclusion of the other six modes was superfluous and their removal from the calculation should not materially alter any conclusions made. The effect of the internal wave on the surface spectrum is, therefore, primarily local in k-space, its width being approximately 2K where K is the internal mode wavenumber.

In linear theory for the case of resonance between the primary surface wave and internal wave, the surface modulation should be stationary with respect to the internal wave. We

FIGURE 2.



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$|q_k| \equiv$ MODULUS OF SURFACE SLOPE
k_o
k_o+K
k_o-K
k_o+2K
k_o-2K
k_o+3K
0
10
20
30
40
50
60
70
SECONDS

may characterize the interaction as Lewis (1971) did in his experiment, by defining a quantity called the surface distortion (S),

$$S = \frac{|G|_{max} - |G|_{min}}{|G|_{max} + |G|_{min}} \tag{4.11}$$

which using the linear calculation in this section reduces to

$$S = U_o \; Kt \left[\sin K\xi + \frac{\Omega t}{8} \cos K\xi \right] \quad . \tag{4.12}$$

Because we have a resonance condition, a given segment of the surface wave interacts continuously with a given phase point of the internal wave as they both propagate down the tank. The length of time of the interaction can therefore be measured as a function of distance down the tank, i.e., $t = x/C_I$. Keeping this in mind and recalling Zachariasen's calculation of the maximum surface distortion, we rewrite Eq. (4.12) as,

$$S_{max} = \sqrt{2} \; S_{rms} = \left(\frac{U_o}{C_I}\right) Kx \sqrt{1 + \frac{(Kx)^2}{64}} \quad . \tag{4.13}$$

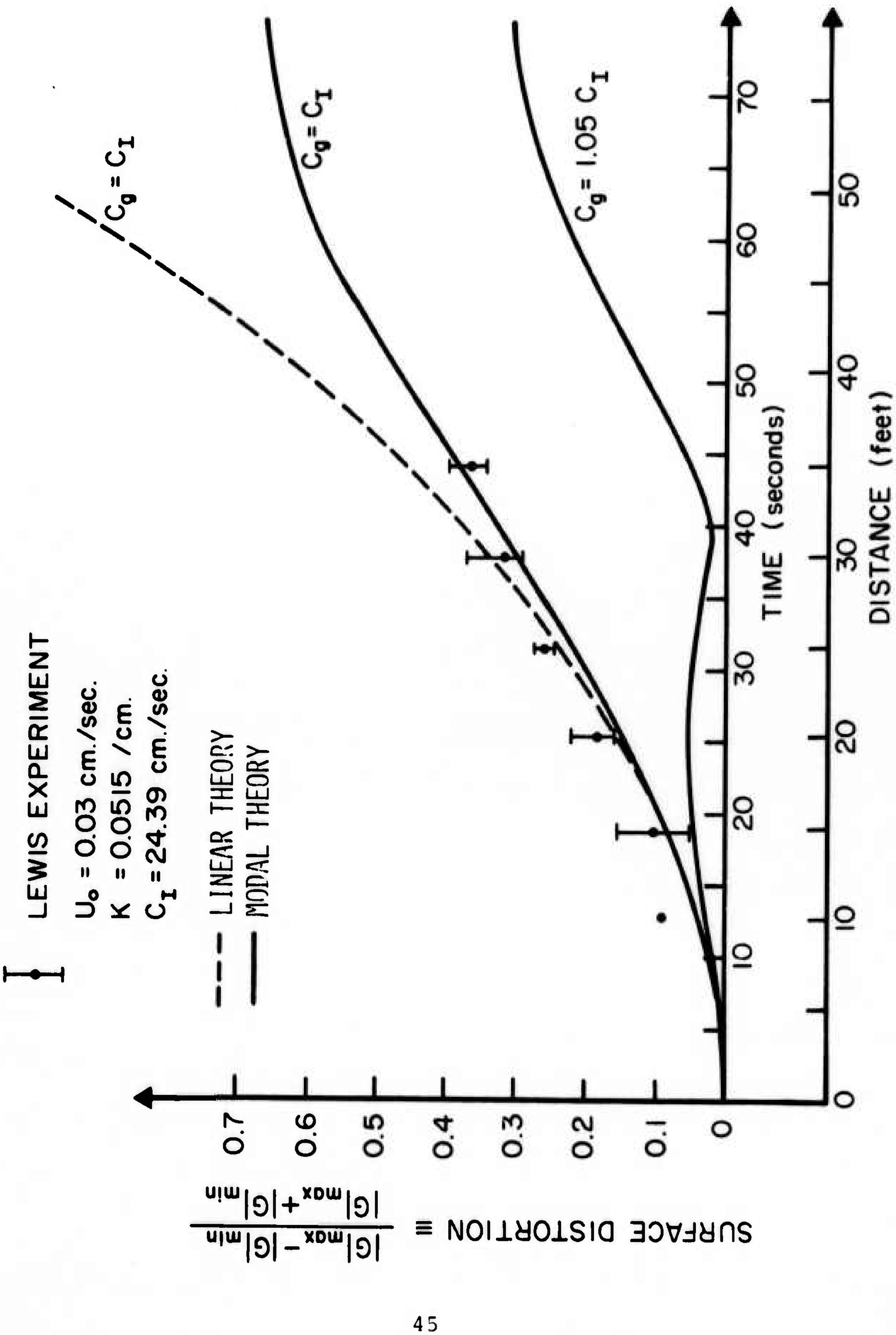
This expression differs in the initial slope from the one given by Zachariasen, but as he pointed out the problem he

considered in detail was the boundary value problem whereas Eq. (4.13) corresponds to the initial value problem. Eq. (4.13) does agree with the linear calculation of Ko (1971) and also the initial value problem of Zachariasen, the solution of which he left in general form.

In Figure (3) we compare the results of the numerical calculation of the interaction with the linear expression in Eq. (4.13) and the results of the Lewis experiment (1971). The linear theory agrees with the numerical calculation for a surprisingly long time. One should recall, however, that the parameters only restrict the interaction time to be less than 50 seconds for the linear case. Near this time the two results diverge, the linear case growing quadradically and the nonlinear numerical calculation saturating to a surface distortion of approximately 0.65. It is unfortunate that for the experimental parameters selected that the tank was not a few feet longer so as to distinguish experimentally between these two calculations.

In the light of these results one could reproduce in the present context the complete analysis given by Ko (1971). The third curve in Figure (3) is an example of such a calculation in which the initial wave is not in resonance with the internal wave and therefore the surface distortion is inhibited. It is not the similarities with

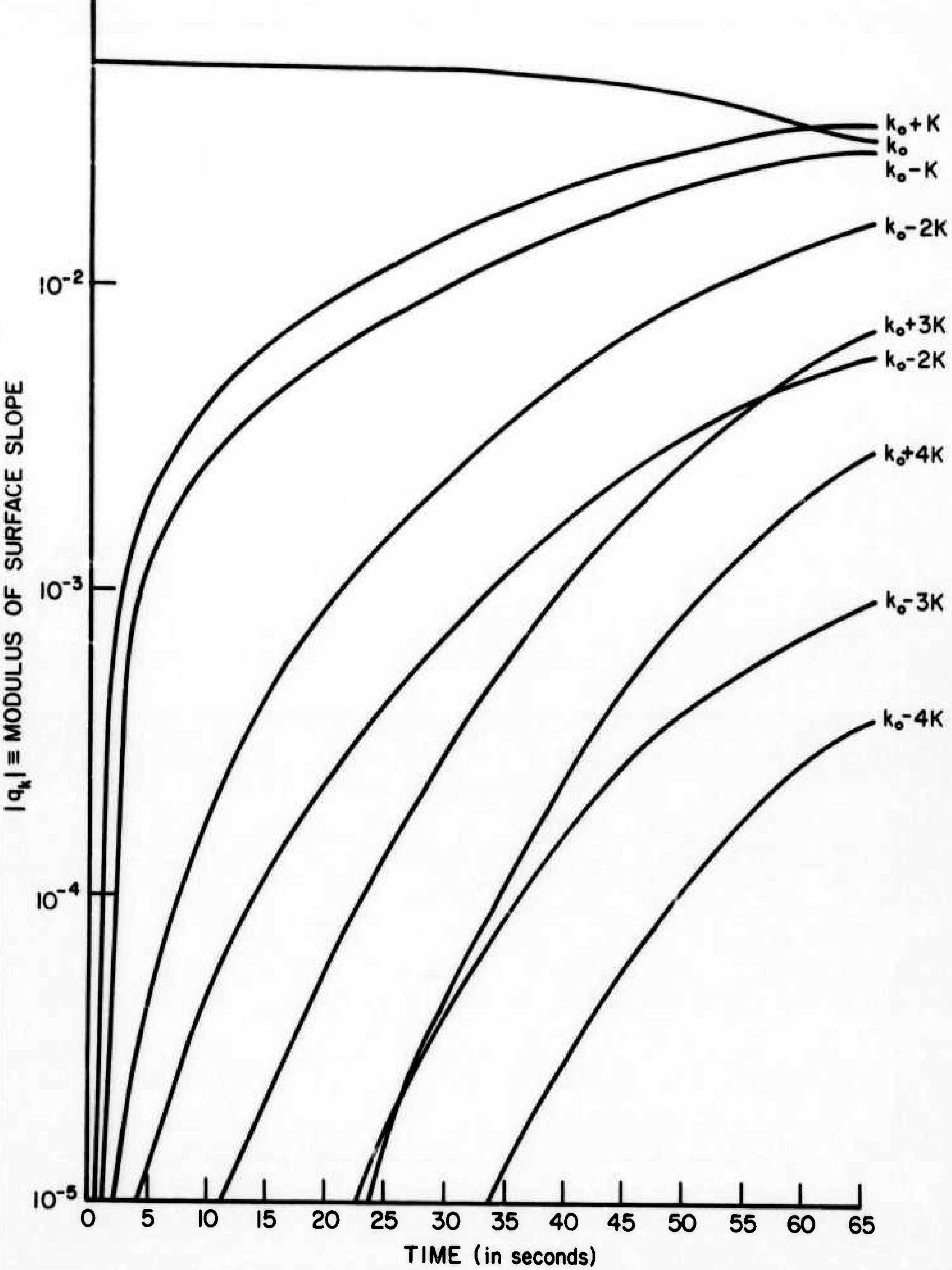
FIGURE 3.


LEWIS EXPERIMENT
U_o = 0.03 cm./sec.
K = 0.0515 /cm.
C_I = 24.39 cm./sec.
LINEAR THEORY
MODAL THEORY
$C_g = C_I$
$C_g = C_I$
C_g = 1.05 C_I
TIME (seconds)
DISTANCE (feet)
SURFACE DISTORTION ≡ $\frac{|G|_{max} - |G|_{min}}{|G|_{max} + |G|_{min}}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
10
20
30
40
50
60
70
0
10
20
30
40
50

linear theory, however, but the differences which are of interest here. In linear theory, the resonant wave and those close to resonance are unique in that the surface distortion grows without limit, whereas the nonresonant waves lead to finite surface distortions. In general, however, if we start with a resonant system the nonlinear interactions will detune the system thereby inducing saturation. The saturation level for the initially resonant system, will, however, be maximum.

In Figure (4) the effect of including the four wave coupling terms in Eq. (3.13) can be seen. The first effect noted is the increased rate of growth in each mode, except of course the primary mode which has an increased rate of attentuation. To understand this effect we recall the discussion in I about the preferential amplification of the Benjamin-Feir sidebands. The sideband frequencies which had maximum amplification were $\omega_{k_{BF}} = \omega_{k_o}(1 \pm q_{k_o})$ where q_{k_o} is the slope of the central wave in our notation. The corresponding wavenumbers are $k_{BF} = k_o(1 \pm 2q_{k_o})$ which for the initial conditions in our problem are $k_{BF} = k_o(1 \pm 0.104)$, i.e., $\pm.043$ cm^{-1} from k_o. It is coincidental that the Benjamin-Feir sidebands lie very close to the sidebands coupled by the internal wave to the primary mode, i.e., $\pm.0515$ cm^{-1} from k_o. This near coincidence of the wavenumber, however,

Figure 4.



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$|q_k| \equiv$ MODULUS OF SURFACE SLOPE
0 5 10 15 20 25 30 35 40 45 50 55 60 65
TIME (in seconds)
k_o+K
k_o
k_o-K
k_o-2K
k_o+3K
k_o-2K
k_o+4K
k_o-3K
k_o-4K

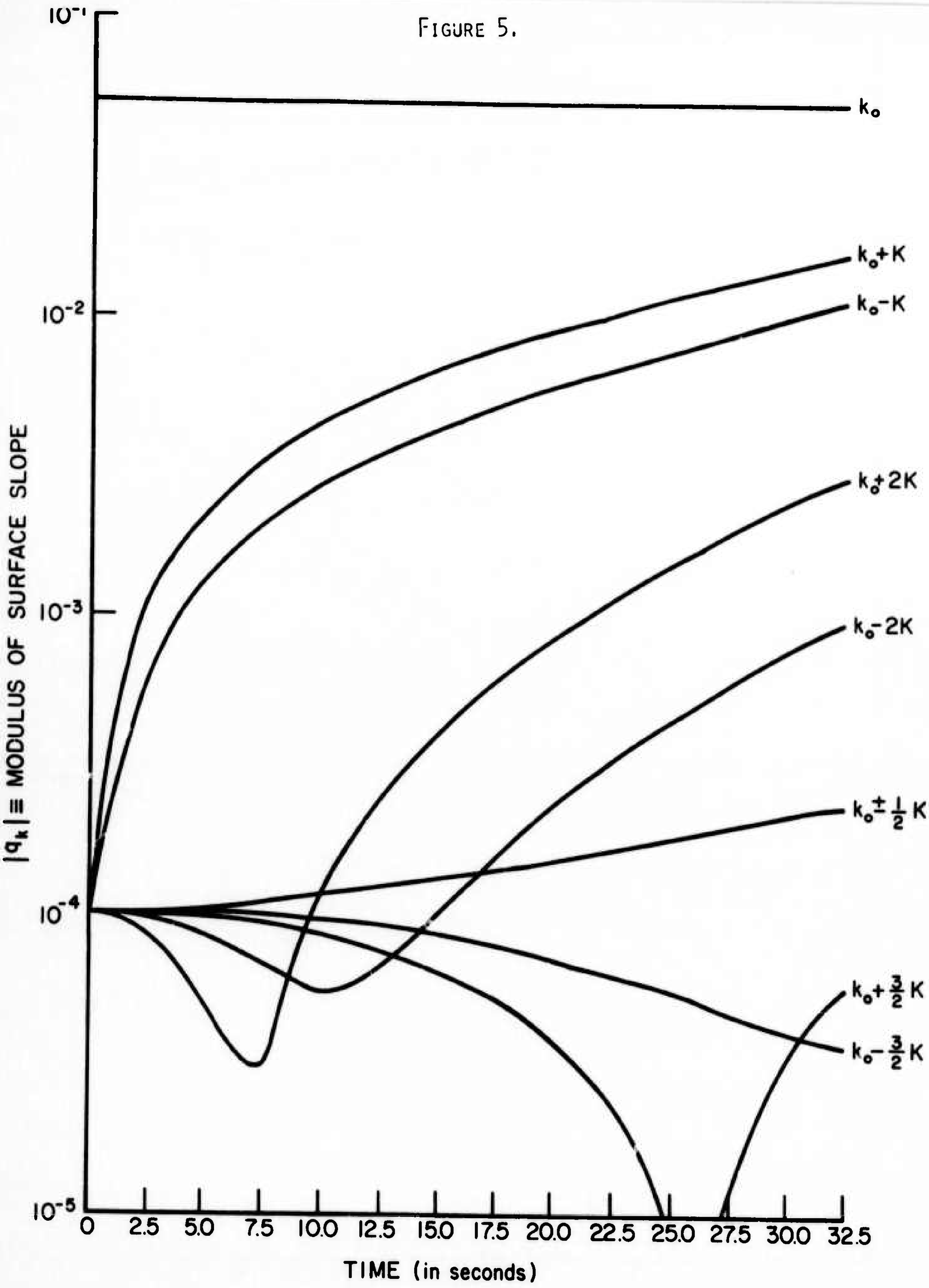


FIGURE 5.
k_o
k_o+K
k_o-K
k_o+2K
k_o-2K
$k_o\pm\frac{1}{2}K$
$k_o+\frac{3}{2}K$
$k_o-\frac{3}{2}K$
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$|q_k| \equiv$ MODULUS OF SURFACE SLOPE
0 2.5 5.0 7.5 10.0 12.5 15.0 17.5 20.0 22.5 25.0 27.5 30.0 32.5
TIME (in seconds)

accounts for the marked increase in the growth rate of the sidebands.

The second effect apparent from comparing Figure (4) with Figure (3) is the loss of oscillation in the mode amplitudes. This occurs because of the change in the phase velocity induced by the nonlinear terms, i.e., a nonlinear dispersion relation. To see this effect in mode detail and to determine if a finer grid for the wavenumbers would significantly change the spectral growth indicated in Figure (4), we introduced modes at $k_o \pm \frac{1}{2}K$ and $k_o \pm \frac{3}{2}K$ in the present problem.

In Figure (5) the growth of the nine modes in the latter calculation is indicated. The initial amplitudes for this calculation are slightly different from the preceeding but this difference should be of no consequence. Of importance to the early time behavior is the initial choice of phases for the modes. We see that with the phases selected the sidebands at $k_o \pm 2K$ initially give up energy to the system. This trend is soon reversed, however, so that at late times ($t > 25$ sec) the growth of the modes directly coupled to the internal wave is not noticeably affected by the initial choice of phases. We also see that the modes at half-odd multiples of K which only grow by means of the nonlinear interaction, play no part in the spectral development.

Comparing Figures (3), (4) and (5) we see that: (i) the energy transferred directly from the internal wave to the surface spectrum is localized in k-space, (ii) the four wave interaction spreads this energy out, i.e., broadens the effect of (i), and (iii) the major effect of the four wave interactions is between waves directly coupled to the primary wave via the internal wave due to the coincidence of the Benjamin-Feir spacing (.043 cm^{-1}) and the wavenumber spacing (.0515 cm^{-1}).

4C. Ocean Spectral Development

In this calculation we wish to contrast the spectral development in an ocean environment with that just calculated for a tank. We consider an impulsively-applied interaction between an internal wave in the ocean and a surface spectrum. We assume the internal wave to be $\pi\times10^2$ meters long; i.e., $K = 2\times10^{-4}$ cm^{-1}, with frequency $\Omega = 10^{-2}$ sec^{-1} so that its phase velocity is $C_I = 50$ cm/sec. The central wavenumber of the surface spectrum is assumed to be in resonance with the internal wave and therefore has a wavelength of 64.18 cm; i.e., $k_o \approx 9.79\ 10^{-2}$ cm^{-1}. The amplitude of the central mode is 1 cm, so that its slope (q_k) is 0.1.

The estimate in Section 4B indicates that the coupling of the central mode to the rest of the spectrum in the ocean is much greater than in the tank case. This implies that the energy will drain much faster from the central mode. For this reason the present calculation is done with 21 modes so as to avoid any difficulty with the finite size of the spectrum in determining the characteristics of the spectral development. To simulate the ocean environment, we assume the remaining twenty modes to have an initial amplitude of 10^{-3}; i.e., two orders of magnitude smaller than the primary and to have random phases.

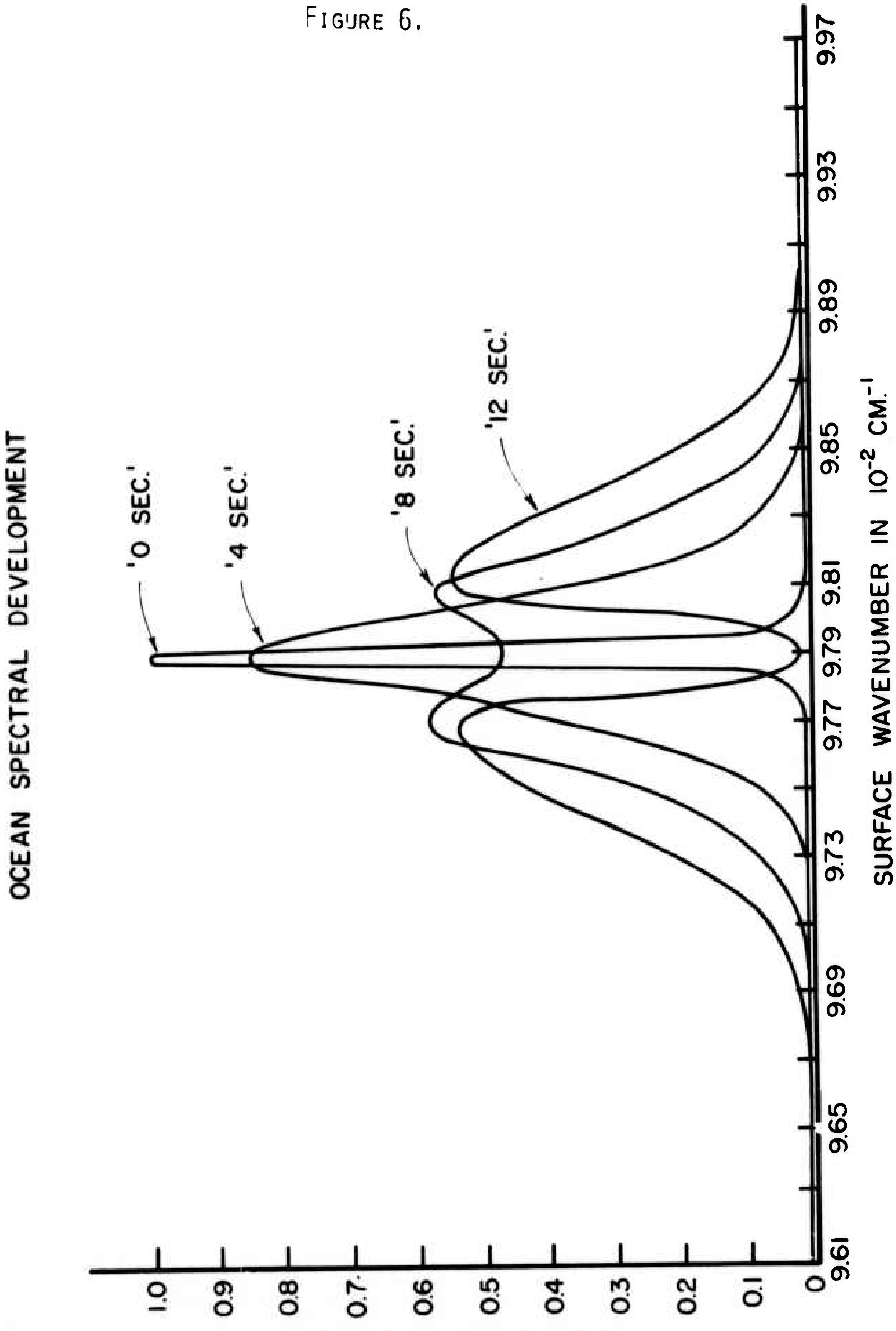
Due to the large number of modes in this calculation, the representation of the results is slightly different than in the tank case. If we omit the four wave coupling, the spectrum is shown in Figures (6) and (7) at different times. Although the number of modes is discrete, in Figure (6) we join the mode amplitudes at a given time to aid the eye in determining the spectrum. The "0 sec" spectrum is the initial state of the problem and is depicted as a very narrow spectrum. After the interaction is "turned on" the spectrum is seen to broaden and develop a bi-modal character.

Figure (7) depicts a continuation of the spectral development initiated in Figure (6). It can be seen that the energy is being symmetrically transferred to both higher and lower wavenumbers as the interaction persists. It is evident that the more central mode amplitudes are oscillating in time and as the bulk of the energy is transferred to the spectral extremes the energy in the central part of the spectrum is decaying. Indeed, if we look at the linear part of the solution to this problem as given by Equation (3.52), we have

$$q_n(t) \approx q_N(o)\ J_{N-v}(\tau) \quad .$$

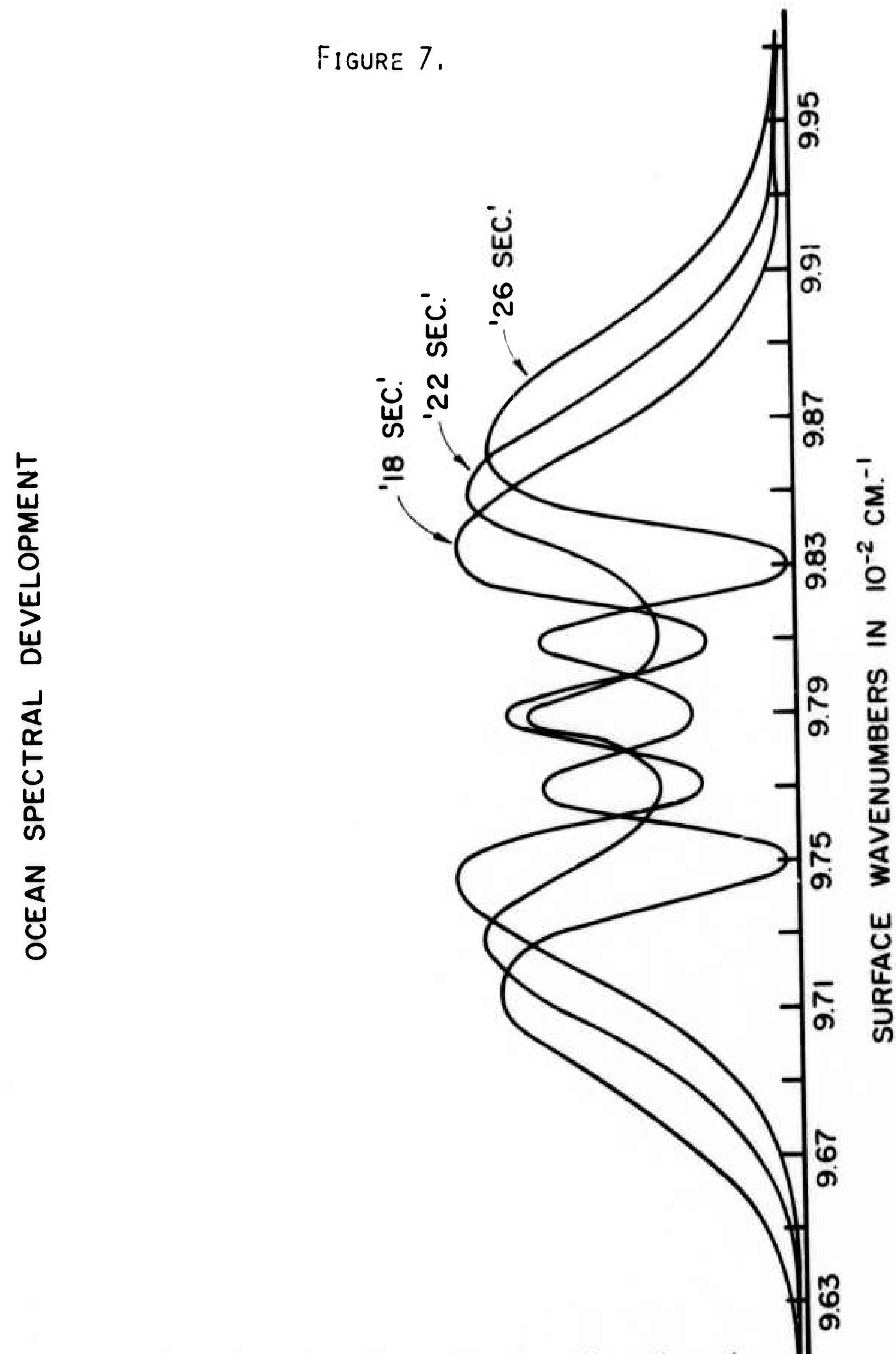
Then using the asymptotic form of the Bessel function, the central part of the spectrum decays as $1/\sqrt{t}$ and oscillates

FIGURE 6.



OCEAN SPECTRAL DEVELOPMENT
'0 SEC.'
'4 SEC.'
'8 SEC.'
'12 SEC.'
MODULUS AMPLITUDE x 10^{-2}
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
9.61
9.65
9.69
9.73
9.77
9.79
9.81
9.85
9.89
9.93
9.97
SURFACE WAVENUMBER IN 10^{-2} CM.$^{-1}$

FIGURE 7.


OCEAN SPECTRAL DEVELOPMENT
'18 SEC.'
'22 SEC.'
'26 SEC.'
MODULUS AMPLITUDE x 10-2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
9.63
9.67
9.71
9.75
9.79
9.83
9.87
9.91
9.95
SURFACE WAVENUMBERS IN 10-2 CM.-1

with a period $T = \frac{1}{k_o U_o}$ which is approximately 5 seconds in our problem. This is the structure that is beginning to appear in Figure (7). The central mode, whose frequency is $\omega_k = 9.79$ rad/sec, has a period of 0.641 sec and is depleted of energy in approximately eighteen cycles. A somewhat lesser amount of energy is then returned to the mode in a somewhat shorter time.

Figure (8) depicts a further contrast with the tank experiment. In this figure we show the surface distortion of the ocean for the 21-mode calculation. If we examine the expression for the modulation function of the surface slope given by Eq. (3.61), retaining only the linear terms we have

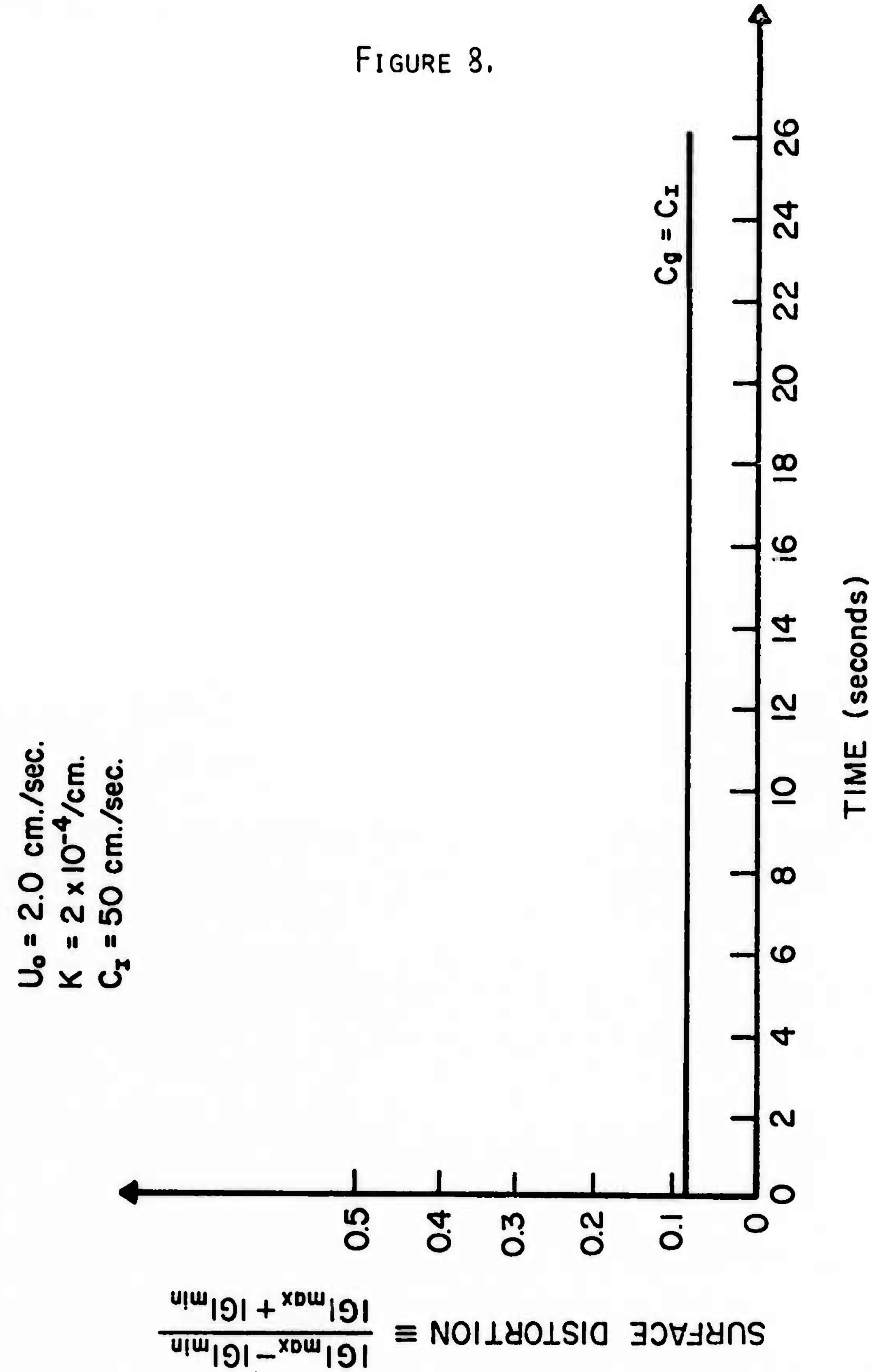
$$G_s(x,t) = \sum_{\nu=-A}^{A} q_N(o)\ J_\nu(\tau)\ z^\nu \tag{4.14}$$

where $z = e^{i(K\xi+\frac{\pi}{2})}$, 2A is the spectral width and $N > A >> 1$. We may approximate the sum in Eq. (4.14) by,

$$\sum_{\nu=-A}^{A} z^\nu\ J_\nu\ (\tau) \approx \exp\left\{i\ \frac{\tau}{2}\left(z - \frac{1}{z}\right)\right\} \tag{4.15}$$

which is the generating function for Bessel functions. Using the definition of z and Eq. (4.15) we obtain for the slope modulation function,

FIGURE 8.



U_o = 2.0 cm./sec.
K = 2 x 10^{-4}/cm.
C_I = 50 cm./sec.
$C_g = C_I$
0 2 4 6 8 10 12 14 16 18 20 22 24 26
TIME (seconds)
0 0.1 0.2 0.3 0.4 0.5
SURFACE DISTORTION $\equiv \frac{|G|_{max} - |G|_{min}}{|G|_{max} + |G|_{min}}$

$$G_s(x,t) \approx q_N(o) \exp\left[i\tau \cos K\xi\right] \quad . \tag{4.16}$$

Because Eq. (4.16) has modulus unity, it is clear that to first order both the surface distortion and slope distortion are constant. This linear result is in agreement with the numerical calculation indicated in Figure (8).

For the initial conditions of this problem, the four wave interaction terms do not affect the qualitative results shown. In fact, for the first 14 seconds or approximately twenty periods of the primary wave there is essentially no quantitative difference in the surface distortion. Further calculations using more "physical" initial conditions will be conducted elsewhere.

4D. Conclusions

It is clear from the calculations discussed in Section 4B and 4C that the ocean and tank experiments fall into two quite different interaction regimes. This is most clearly illustrated by the growth of the surface distortion found in the tank experiment and calculation depicted in Figure (4) and the complete lack of growth found under oceanographic conditions for an equivalent interaction depicted in Figure (8). This result points up the extreme caution required in the design of tank experiments whose purpose is to induce information about interactions on the real ocean.

The calculations clearly demonstrate the need for detailed analysis of experiments to determine their applicability to the ocean environment. To this end a program of calculations is being initiated which will determine spectral modifications produced by surface currents under a variety of oceanographic conditions. Calculations will also be made for the corresponding tank experiments where such experiments can be conducted. An example of such a calculation would be that of a stationary current pattern in the lab which simulates the convergence zone of a long internal wave. Since such a simulated internal wave could

be constructed by contouring the bottom of a tank appropriately, the difficulty encountered in generating a translating internal wave in a tank is avoided.

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